

# A NEW PROOF FOR THE BANACH-ZARECKI THEOREM: A LIGHT ON INTEGRABILITY AND CONTINUITY

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**ABSTRACT.** To demonstrate more visibly the close relation between the continuity and integrability, a new proof for the Banach-Zarecki theorem is presented on the basis of the Radon-Nikodym theorem which emphasizes on measure-type properties of the Lebesgue integral. The Banach-Zarecki theorem says that a real-valued function  $F$  is absolutely continuous on a finite closed interval if and only if it is continuous and of bounded variation when it satisfies Lusin's condition. In the present proof indeed a more general result is obtained for the Jordan decomposition of  $F$ .

## 1. Introduction

The original motivation for the present work concerns with the open debate of the regularity of hydrodynamical parameters of fluid flows. It is still not known that starting from a smooth initial conditions in a three dimensional fluid, when and how any kind of blow up or singularity will happen. A large amount of works consider this problem in various special cases and obtain many results. It was known that the type of singularity is so strong such that many kinds of integral norms of hydrodynamical quantities are also singular. However, almost all of these integral norms

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MSC(2010): Primary: 26B30; Secondary: 46B22, 46G12.

Keywords: Banach-Zarecki theorem, Radon-Nikodym theorem, Lusin's condition.

Received: 20 September 2011, Accepted: 3 June 2012.

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are obtained by the Lebesgue integration but we know that there are other types of integration that are generalizations of the usual Riemann integral and do not coincide the Lebesgue integral.

So, a natural question comes that how can we say something when our functions are not Lebesgue integrable? How should one replace (absolute) continuities and regularities in these new cases? As the first step it looks necessary to test and generalize a direct relation between the integration and continuity and Banach–Zarecki Theorem provides perhaps the most visible case to observe such a relation. It was therefore needed to discover a more direct and closer relation between the absolute continuity and the Lebesgue integral to be an arrow for other works.

Banach–Zarecki Theorem is a classical theorem in real analysis with many applications mostly in geometric and functional analysis as well as some physical and engineering subjects. The origin of this theorem was stated and proved by Banach and independently by Zarecki for a real-valued function on an interval [10]. For functions of a real variable with values in reflexive Banach spaces, the result is contained in [6], Theorem 2.10.13, where the codomain space has the Radon-Nikodym property. There also exists another version of the theorem initiated by an old result of Lusin [8], later extended for a function of a real variable with values in a metric space [3, 4].

It is not surprising that there is a variety of extensions for this theorem to more variables in many ways and also by natural changes in properties well-known in one dimensional case such as almost everywhere continuity and differentiability, integration by parts and so on [5, 12, 9]. In fact this theorem can be generalized to the concept of approximate continuity that plays an important role to understand the relationship between Riemann integrability (for almost everywhere continuous functions) and continuity on the one hand, and the relationship between approximate continuity and Lebesgue integrability (for almost everywhere approximately continuous functions), on the other hand [4].

There exist alternative proofs for this theorem; although these are of different appearance but they are constructed from a common root (see e.g. [1, 2, 11, 13]). In the present work the classical form of the theorem is considered, since it looks possible to naturally extend the results to more general cases mentioned above. The most convenient statement of the Banach–Zarecki theorem is [1]:

**Theorem 1.1.** *Let  $F$  is a real-valued function defined on a real bounded closed interval  $[a, b]$ . A necessary and sufficient condition for  $F$  to be*

*absolutely continuous is that*

- (i)  *$F$  is continuous and of bounded variation on  $[a, b]$ ,*
- (ii)  *$F$  satisfies Lusin's condition, i.e. it maps sets of Lebesgue measure zero into sets of Lebesgue measure zero.*

The necessary condition is straightforward and will not be discussed here. Its proof is given in almost any text book of real analysis [1, 7]. However the sufficient condition is rather technical and requires some non-trivial efforts and may rarely be found in common references. Thus, our attempt is concentrated on providing an alternative proof of the sufficient condition, that is, if a real-valued function is continuous and of bounded variation and also satisfies Lusin's condition, then it is absolutely continuous. In [1], there is a proof for the sufficient condition employing an inequality being also proved in this reference. The main tools of this approach are the almost everywhere differentiability and the Vitali covering theorem.

However the present proof is based on the close relation between the Lebesgue integral and the properties of a measure space which manifests itself essentially through the Radon-Nikodym theorem. Thus, the main used tools here are the Radon-Nikodym theorem and the properties of variations of functions. This new proof may however cost to be considered because of several reasons such as the following. Here a slightly more general result is proven, namely Lemma 2.2 while we need only Corollary 2.3 for our proof. The concept of almost everywhere differentiability and thus the Vitali covering lemma is not used. The methods and techniques handled here seem to be applicable and naturally generalizable to a class of similar problems. There is a hope to generalize this method to obtain an analog version for the absolute continuity in relation with other types of integration rather than the Lebesgue integral.

Finally it is seen that here some statements are proven employing only conditions (i) and (ii) mentioned in the Banach-Zarecki theorem and without using the absolute continuity condition, while these statements are usually proved through a direct application of the absolute continuity condition in the common literatures.

In order to prove Theorem 1.1, our strategy is to establish the following theorem which illustrates more clearly, the relation between the absolute continuity and the Lebesgue integral.

**Theorem 1.2.** *Suppose that  $F : [a, b] \rightarrow \mathbb{R}$  is a continuous and of bounded variation and satisfies Lusin's condition. Then there exists an integrable function and in fact a Borel-measurable function  $f : [a, b] \rightarrow$*

$\mathbb{R}$  such that

$$F(x) = F(a) + \int_{[a,x]} f d\lambda : \forall x \in [a, b],$$

where  $d\lambda$  in the integral comes from the Lebesgue measure  $\lambda$ .

This theorem will immediately yield Theorem 1.1 through the application of the well known statement [1, 7]:

*Let  $f : [a, b] \rightarrow \mathbb{R}$  be a Lebesgue integrable function and let  $F(x) = F(a) + \int_{[a,x]} f d\lambda$ , then  $F$  is absolute continuous on  $[a, b]$ .*

In the next section, we prove the Theorem 1.2 in three steps, the first of which is well known in text books [7] while step 2 and especially step 3 are of our main interests.

Throughout this paper we assume that the notation  $\lambda$  implies the Lebesgue measure, unless specially stated otherwise.

## 2. The main result: new proof of Theorem 1.2

The proof is divided into three interconnected steps.

*Step 1.* At first, we prove the theorem assuming that  $F$  is strictly increasing. In this case, the proof coincides the standard proof given in common text books (see e.g. Theorem 4.3.8 of [7]) which employs the Radon–Nikodym theorem. To have a complete discussion, let us briefly review the proof here.

Since  $F$  is strictly increasing,  $F$  is a homeomorphism from  $I = [a, b]$  to  $J = F(I) = [F(a), F(b)]$  and so  $F$  preserves Borel sets between  $I$  and  $J$ . Let  $\mathcal{B}$  be the collection of Borel measurable subsets of  $I$ , then we can define the new measure  $\nu : \mathcal{B} \rightarrow [0, \infty)$  as  $\nu(E) = \lambda(F(E))$ . It is clear that  $\nu$  is a finite measure and is absolutely continuous relative to  $\lambda$  (since  $F$  satisfies Lusin's condition). Therefore, according to the Radon–Nikodym theorem, there exists a (Borel) measurable and Lebesgue integrable function  $f : I \rightarrow \mathbb{R}$  such that

$$(2.1) \quad \nu(E) = \int_E f d\lambda, \quad E \in \mathcal{B}.$$

Especially if  $E = [a, x]$  for  $x \in I$ , then  $F(E) = [F(a), F(x)]$  and Eq. (2.1) immediately implies that

$$F(x) = F(a) + \int_{[a,x]} f d\lambda, \quad x \in I.$$

This completes the proof of this step.

*Step 2.* Let  $F$  is non-decreasing (i.e. increasing but not strictly increasing). So, there exists the continuous and of bounded variation function  $G(x) = F(x) + x$  which is strictly increasing. The proof will be complete if we prove that  $\lambda(N) = 0$  then  $\lambda(G(N)) = 0$ . Since  $F$  is non-decreasing, one easily observes that the constant values of  $F$  make sense in disjoint intervals  $S_k$  and the continuity of  $F$  implies that  $S_k$ s are closed intervals, say  $[a_k, b_k]$ . Hence, in general, on  $S = \bigcup_{k=1}^{+\infty} S_k$ ,  $F$  takes the values  $F(S) = \left\{ \mu_k \right\}_{k=1}^{+\infty}$  where  $\mu_k$  is the value of  $F$  on  $S_k$ .

The intervals  $S_k$  may be so small and their union  $S$  is not necessary closed. Now, since  $S_k$  s are disjoint, we can write

$$N_1 = N \cap S, \quad N_2 = N - N_1.$$

Therefore, we have

$$\lambda(G(N)) \leq \lambda(G(N_1)) + \lambda(G(N_2)),$$

while

$$\lambda(G(N_1)) = \lambda\left(\bigcup_{k=1}^{+\infty} G(N \cap S_k)\right) \leq \sum_{k=1}^{+\infty} \lambda(G(N \cap S_k)).$$

On the other hand,  $G(N \cap S_k) = \{\mu_k + x \mid x \in N \cap S_k\}$  and thus  $\lambda(G(N \cap S_k)) = \lambda(N \cap S_k)$ , so

$$\begin{aligned} \lambda(G(N_1)) &\leq \sum_{k=1}^{+\infty} \lambda(G(N \cap S_k)) \\ &= \lambda\left(\bigcup_{k=1}^{+\infty} (N \cap S_k)\right) = \lambda(N_1) \leq \lambda(N) = 0. \end{aligned}$$

Therefore  $\lambda(G(N_1)) = 0$ . To prove  $\lambda(G(N_2)) = 0$ , we notice that  $F$  satisfies  $\lambda(N_2) = 0$  results in  $\lambda(F(N_2)) = 0$ , so for each  $\epsilon > 0$ , we can find an open set  $U$  such that  $F(N_2) \subset U$  with  $\lambda(U) < \epsilon$ . In addition, since  $\lambda(N_2) = 0$ , one can find an open set  $U'$  including  $N_2$

such that  $\lambda(U') < \epsilon$ . The open set  $V := U' \cap F^{-1}(U)$  contains  $N_2$  such that  $\lambda(V) < \epsilon$  and  $\lambda(F(V)) < \epsilon$ . Suppose  $V = \bigcup_{k=1}^{+\infty} I_k$  where  $I_k$ s are disjoint open intervals. For each  $I_k$ , consider the two closed intervals (if exist)  $S_i$  and  $S_j$  intersecting  $I_k$  from the left and right containing the left and right boundary points of  $I_k$  resp. Define  $I'_k = I_k - (S_i \cup S_j)$  (it is possible that  $I'_k$  is empty). Thus  $I'_k \subset I_k$  and  $I'_k$ s are mutually disjoint.

Let  $V' := \bigcup_{k=1}^{+\infty} I'_k$  and since  $S_k$ s are all out of  $N_2$ ,  $N_2 \subset V'$  and thus  $F(N_2) \subset F(V')$ . It is important to attend that for each  $l$  and  $k$ ,  $S_l$  is either completely contained in  $I'_k$  or is disjoint from it. According to the conditions on  $F$ , i.e. non-increasing and continuity, one can deduce that  $F(I'_k)$  is an interval (not necessarily closed or open) which we denote it by  $J_k$ . Now we acclaim that  $J_k$ s are mutually disjoint. If else, for example if  $y \in J_k \cap J_l$  for some  $k$  and  $l$ , then there exist at least two points  $x_k \in I'_k$  and  $x_l \in I'_l$  such that  $F(x_k) = F(x_l)$ . Hence there exists an  $S_i$  so that  $[x_k, x_l] \subset S_i$  but hence  $S_i$  is not completely in  $I'_k$  or  $I'_l$  which is a contradiction. Relations

$$F(N_2) \subset F(V') = \bigcup_{k=1}^{+\infty} F(I'_k) \subset U,$$

imply that

$$\lambda\left(\bigcup_{k=1}^{+\infty} J_k\right) \leq \lambda(U) < \epsilon,$$

and since  $J_k$ s are disjoint sets,

$$\sum_{k=1}^{+\infty} \lambda(J_k) < \epsilon.$$

The remaining work is to determine  $G(I'_k)$ s and approximate their measure. For each  $k$ , we have

$$\begin{aligned} G(I'_k) &= \left\{ F(x) + x \mid x \in I'_k \right\} \\ &\subseteq \left\{ y + x \mid y \in J_k, x \in I'_k \right\} \\ &\subseteq \left( \inf(I'_k) + \inf(J_k), \sup(I'_k) + \sup(J_k) \right), \end{aligned}$$

which results in

$$\lambda(G(I'_k)) \leq \lambda(I'_k) + \lambda(J_k).$$

So we obtain

$$\lambda(G(N_2)) \leq \lambda\left(\bigcup_{k=1}^{+\infty} G(I'_k)\right) \leq \sum_{k=1}^{+\infty} \lambda(G(I'_k)).$$

The latter equations clarify that

$$\lambda(G(N_2)) \leq \sum_{k=1}^{+\infty} \lambda(I'_k) + \sum_{k=1}^{+\infty} \lambda(J_k) < \epsilon + \lambda(U) < 2\epsilon.$$

Thus  $\lambda(G(N_2)) = 0$ . This shows that Lusin's condition is fulfilled for  $G(x) = F(x) + x$ . Now pertaining to Step 1, there is an integrable and Borel-measurable function  $f_1 : [a, b] \rightarrow \mathbb{R}$  s.t.

$$G(x) - G(a) = \int_{[a,x]} f_1 d\lambda,$$

hence

$$F(x) - F(a) = \int_{[a,x]} f d\lambda,$$

thus if we let  $f = f_1 - 1$  and this completes the proof of Step 2.

*Step 3.* Finally, we assume that  $F$  is continuous and of bounded variation which satisfies Lusin's condition and show that the theorem holds. To accomplish this, we make use of the following two lemmas.

**Lemma 2.1.** *Let  $F : [a, b] \rightarrow \mathbb{R}$  be a continuous function of bounded variation. If  $F = p - n$  is the Jordan decomposition for  $F$ , then  $p$  and  $n$  are continuous.*

**Lemma 2.2.** *In the situation of Lemma 2.1, let  $N \subset [a, b]$  such that  $F(N)$  has a zero Lebesgue measure. Then  $p(N)$  and  $n(N)$  are also of Lebesgue measure zero.*

Lemma 2.2 immediately yields the following result.

**Corollary 2.3.** *In the situation of Lemma 2.1, let  $F$  satisfies Lusin's condition, then  $p$  and  $n$  also satisfy Lusin's condition.*

It is seen from Lemma 2.1 and Corollary 2.3 that both of  $p$  and  $n$  are continuous and of bounded variation and Lusin's condition is valid for them. Then since they are non-decreasing, there will exist integrable and Borel-measurable real-valued functions  $g$  and  $h$  on  $[a, b]$  so that  $p(x) = p(a) + \int_{[a,x]} g d\lambda$  and  $n(x) = n(a) + \int_{[a,x]} h d\lambda$  and therefore the proof will be completed substituting  $f = g - h$ .

### 3. Proof of Lemma 2.1

It is sufficient to prove that  $p$  is continuous. At first, we denote that  $p$  is a right-continuous function. The continuity of  $p$  can be directly achieved by  $(\epsilon - \delta)$  method. However an alternative proof is presented here because of its easier application in the proof of Lemma 2.2.

According to definition,

$$(3.1) \quad p(x) = \bigvee_a^x (F) = \sup_P |F(P)| = \sup_P \sum_{k=1}^{n(P)} |F(x_k) - F(x_{k-1})|$$

is the variation of  $F$  from  $a$  to  $x$  where the supremum is taken over all partitions

$$P : a = x_0 < x_1 < \cdots < x_n = x$$

of  $[a, x]$  and  $n = n(P) = \#P - 1$ . Therefore for arbitrary  $\epsilon > 0$  there is a partition  $P$  such that

$$(3.2) \quad 0 \leq \bigvee_a^x (F) - |F(P)| < \epsilon.$$

**Definition 3.1.** For the given partition  $P : a = x_0 < x_1 < \cdots < x_n = b$ , let  $x \in [x_{i-1}, x_i]$ . Two adjacent partitions  $P_1(x)$  and  $P_2(x)$  are defined as

$$\begin{aligned} P_1(x) &: a = x_0 < \cdots < x_{i-1} \leq x, \\ P_2(x) &: x \leq x_i < \cdots < x_n = b, \end{aligned}$$

and partition  $P'(x)$  considered as a refinement of  $P$  is

$$P'(x) : a = x_0 < \cdots < x_{i-1} \leq x \leq x_i < \cdots < x_n = b.$$

For  $\epsilon > 0$  and its corresponding partition  $P$  considered in Eq. (3.2), one can define continuous functions  $w_i : [x_{i-1}, x_i] \rightarrow \mathbb{R}$  as

$$(3.3) \quad w_i(x) = |F(P_1(x))|.$$

Application of the pasting lemma implies the existence of the continuous function  $u_\epsilon : [a, b] \rightarrow \mathbb{R}$  so that on each  $[x_{i-1}, x_i]$ ,  $u_\epsilon$  is equal to  $w_i$ . Therefore

$$\begin{aligned} \left( \bigvee_a^x (F) - |F(P_1(x))| \right) + \left( \bigvee_x^b (F) - |F(P_2(x))| \right) &= \bigvee_a^b (F) - |F(P'(x))| \\ &< \epsilon. \end{aligned}$$



The two terms on the left hand side of the above relation are nonnegative and especially considering the first term, one finds that

$$(3.4) \quad 0 \leq p(x) - u_\epsilon(x) < \epsilon,$$

in which Eqs. (3.2) and (3.3) were applied.

Now consider  $\left\{u_{2^{-k}}\right\}_{k=1}^{\infty}$  as a sequence of continuous functions. Equation (3.4) with  $\epsilon = 2^{-k}$  shows that this sequence converges uniformly to  $p$ , thus  $p$  is continuous.

#### 4. Proof of Lemma 2.2

Let  $N \subset [a, b]$  such that  $\lambda(F(N)) = 0$ . For arbitrary  $\epsilon > 0$ , consider its corresponding partition  $P$  as introduced in Eq. (3.2). It is sufficient to prove that  $\lambda(p(N_i)) = \lambda(n(N_i)) = 0$  where  $N_i = N \cap [x_{i-1}, x_i]$  ( $1 \leq i \leq n$ ). Since  $F(N_i)$  has zero Lebesgue measure, there exists a sequence of disjoint open intervals  $\{J_k\}_{k=1}^{\infty}$  such that  $F(N_i) \subset \bigcup_{k=1}^{\infty} J_k$  and

$$(4.1) \quad \sum_{k=1}^{\infty} \lambda(J_k) < \epsilon.$$

At most one of  $J_k$  s contains the point  $F(x_{i-1})$  and at most one of them contains  $F(x_i)$ . If so, we exclude these two points from  $J_k$  s and split the interval(s) containing the points into two adjacent open intervals. This process clearly leaves relation (4.1) unchanged. For each  $J_k$  we have  $F^{-1}(J_k) = \bigcup_{l=1}^{\infty} I_{kl}$  where intervals  $I_{kl} = (a_{kl}, b_{kl})$  are disjoint. According to our hypothesis, one can easily observe that

$$(4.2) \quad \lambda(p(N_i)) \leq \sum_{k,l=1}^{\infty} \lambda(p(I_{kl})).$$

Choose any finite number of intervals  $I_{kl}$ s and call them  $(a_1, b_1), \dots, (a_m, b_m)$  in such an order that we have the partition

$$(4.3) \quad Q : b_0 = x_{i-1} \leq a_1 < b_1 < a_2 < \dots < a_m < b_m \leq a_{m+1} = x_i.$$

Thus

$$\bigvee_{x_{i-1}}^{x_i} (F) - |F(Q)| < \epsilon,$$

which means that

$$\sum_{j=1}^m \left( \bigvee_{a_j}^{b_j} (F) - |F(b_j) - F(a_j)| \right) + \sum_{j=0}^m \left( \bigvee_{b_j}^{a_{j+1}} (F) - |F(a_{j+1}) - F(b_j)| \right) < \epsilon.$$

Each term in the left side is nonnegative, especially noting to the first term and recalling the definition of  $p$  through Eq. (3.1) and its non-decreasing property, one concludes that

$$\sum_{j=1}^m \lambda(p(a_j, b_j)) < \epsilon + \sum_{j=1}^m |F(b_j) - F(a_j)|.$$

The above inequality holds for any finite number of  $I_{kl}$  s, thus

$$(4.4) \quad \sum_{k,l=1}^{\infty} \lambda(p(I_{kl})) < \epsilon + \sum_{k,l=1}^{\infty} |F(b_{kl}) - F(a_{kl})|.$$

Our next task is to find an upper bound proportional to  $\epsilon$  for the second term of the last equation. To do this we consider two separate cases. The first case is when  $F(x_{i-1}) = F(x_i)$ . Choose again the finite number of  $I_{kl}$  s say  $(a_j, b_j)$  s for  $1 \leq j \leq m$  and construct the partition  $Q$  as introduced in Eq. (4.3). This partition is a refinement of  $x_{i-1} < x_i$  and so  $|F(Q)| - |F(x_i) - F(x_{i-1})| < \epsilon$  and thus

$$\sum_{j=1}^m |F(b_j) - F(a_j)| \leq |F(Q)| < \epsilon.$$

The last inequality holds for any finite number of  $I_{kl}$  s and so is also valid for all of them. Therefore when  $F(x_{i-1}) = F(x_i)$  by the use of Eq. (4.4) we have

$$(4.5) \quad \lambda(p(N_i)) \leq \sum_{k,l=1}^{\infty} \lambda(p(I_{kl})) < 2\epsilon.$$

The second case is related to the condition  $F(x_{i-1}) < F(x_i)$  (the opposite case is similar). Recall that  $J_k$  s where disjoint open intervals containing  $F(N_i)$  (except probably the two points  $F(x_{i-1})$  and  $F(x_i)$ ) with total measure less than  $\epsilon$ . Thus we are able to divide them into three types:  $J_k^+$  s whose points are greater than  $F(x_i)$ ,  $J_k^-$  s whose points are less than  $F(x_{i-1})$  and  $J_k^\circ$  s whose points are between  $F(x_{i-1})$  and  $F(x_i)$ . At first attend to  $J_k^+$  s. In this case, take any finite number of  $I_{kl}^+$  s (whose images are inside  $J_k^+$  s) say  $(a_j^+, b_j^+)$  s for  $1 \leq j \leq m$

such that  $x_{i-1} \leq a_1^+ < b_1^+ < a_2^+ < \dots < a_m^+ < b_m^+ \leq x_i$ . Images of these intervals lie inside a finite (say  $s$ ) number of  $J_k^+$  s, namely  $J_{k_r}^+ = (c_r^+, d_r^+)$  for  $1 \leq r \leq s$  where obviously  $s \leq m$ .

Suppose that in addition,  $(c_r^+, d_r^+)$  s are arranged increasingly such that  $F(x_i) \leq c_1^+ < d_1^+ < c_2^+ < \dots < c_s^+ < d_s^+$ . The compact set  $[x_{i-1}, x_i] \cap F^{-1}(c_1^+)$  has a minimum and maximum respectively  $\alpha^+$  and  $\beta^+$ . Since the images of all  $(a_j^+, b_j^+)$  s are greater than  $c_1^+ \geq F(x_i)$ , the intermediate value theorem implies that they all lie between  $\alpha^+$  and  $\beta^+$ . Thus there exist partition  $R_1 : x_{i-1} < \alpha^+ < \beta^+ < x_i$  and its refinement  $R_2 : x_{i-1} < \alpha^+ < a_1^+ < b_1^+ < \dots < a_m^+ < b_m^+ < \beta^+ < x_i$ . The relation  $|F(R_2)| - |F(R_1)| < \epsilon$  regarding the fact that  $F(\alpha^+) = F(\beta^+) = c_1^+$  implies that

$$\sum_{j=1}^m |F(b_j^+) - F(a_j^+)| < \epsilon,$$

but since this is true for any finite number of considered intervals, so for  $I_{kl}^+ = (a_{kl}^+, b_{kl}^+)$  s we have

$$(4.6) \quad \sum_{k,l} |F(b_{kl}^+) - F(a_{kl}^+)| < \epsilon.$$

Quite similarly, for  $I_{kl}^- = (a_{kl}^-, b_{kl}^-)$  s we have

$$\sum_{k,l} |F(b_{kl}^-) - F(a_{kl}^-)| < \epsilon.$$

Finally consider  $J_k^\circ$  s whose points are between  $F(x_{i-1})$  and  $F(x_i)$  where for each  $k$ ,  $F^{-1}(J_k^\circ) = \bigcup_{l=1}^\infty I_{kl}^\circ$ . Similar to the previous case choose a finite number of  $I_{kl}^\circ$  s such as  $(a_j^\circ, b_j^\circ)$  s for  $1 \leq j \leq m$  such that  $x_{i-1} \leq a_1^\circ < b_1^\circ < a_2^\circ < \dots < a_m^\circ < b_m^\circ \leq x_i$  and assume their images lie in  $J_{k_r}^\circ = (c_r^\circ, d_r^\circ)$  for  $1 \leq r \leq s$  where clearly  $s \leq m$ . Again suppose  $(c_r^\circ, d_r^\circ)$  s are arranged increasingly such that

$$(4.7) \quad F(x_{i-1}) \leq c_1^\circ < d_1^\circ < c_2^\circ < \dots < c_s^\circ < d_s^\circ \leq F(x_i).$$

Now define  $\alpha_r^\circ = \min([x_{i-1}, x_i] \cap F^{-1}(c_r^\circ))$  for  $1 \leq r \leq s$ . Relation (4.7) and the intermediate value theorem establish that

$$(4.8) \quad x_{i-1} \leq \alpha_1^\circ < \alpha_2^\circ < \dots < \alpha_s^\circ < \alpha_{s+1}^\circ = x_i.$$

Note that in the above relation  $\alpha_{s+1}^\circ$  is defined to be  $x_i$ . In addition, define  $\beta_r^\circ = \max([x_{i-1}, \alpha_{r+1}^\circ])$

$\cap F^{-1}(d_r^\circ)$  for  $1 \leq r \leq s$  and also define  $\beta_0^\circ = x_{i-1}$ . This definition immediately yields that for each  $r = 1, \dots, s-1$  we have  $\alpha_r^\circ < \beta_r^\circ < \alpha_{r+1}^\circ$  while for  $r = 0$  we have  $x_{i-1} = \beta_0^\circ \leq \alpha_1^\circ$  and for  $r = s$  we have  $\alpha_s^\circ < \beta_s^\circ \leq \alpha_{s+1}^\circ = x_i$ . Thus, relation (4.8) is finally improved to admit to define the partition

$$(4.9) \quad S_1 : x_{i-1} = \beta_0^\circ \leq \alpha_1^\circ < \beta_1^\circ < \alpha_2^\circ < \dots < \alpha_s^\circ < \beta_s^\circ \leq \alpha_{s+1}^\circ = x_i.$$

In this position we claim that for each  $j, r$  ( $1 \leq j \leq m, 0 \leq r \leq s$ ) we have  $(a_j^\circ, b_j^\circ) \cap (\beta_r^\circ, \alpha_{r+1}^\circ) = \emptyset$ . If not, assume  $y$  belongs to this set, then only two states may happen:

In the first state we have  $F(y) < d_r^\circ < c_{r+1}^\circ$  for  $1 \leq r \leq s-1$ ,  $F(y) < c_1^\circ$  for  $r = 0$  and  $F(y) < d_s^\circ$  for  $r = s$ . The case  $r = 0$  has no sense because  $y \in (a_j^\circ, b_j^\circ)$  and the images of all  $(a_j^\circ, b_j^\circ)$  s are greater than  $c_1^\circ$ . When  $r = s$  since  $F(y) < d_s^\circ \leq F(\alpha_{s+1}^\circ) = F(x_i)$ , the intermediate value theorem implies that there exists a point  $z \in (y, x_i]$  such that  $F(z) = d_s^\circ$ . But according to the definition of  $\beta_s^\circ$  we must have  $z \leq \beta_s^\circ$  which contradicts with the position of  $y$ . Finally when  $1 \leq r \leq s-1$ , since  $F(y) < d_r^\circ < F(\alpha_{r+1}^\circ) = c_{r+1}^\circ$ , the intermediate value theorem implies that there exists a point  $z' \in (y, \alpha_{r+1}^\circ)$  such that  $F(z') = d_r^\circ$  but according to the definition of  $\beta_r^\circ$  we must have  $z' \leq \beta_r^\circ$  which is a contradiction.

On the other hand in the second state we may have  $d_r^\circ < c_{r+1}^\circ < F(y)$  for  $1 \leq r \leq s-1$ ,  $c_1^\circ < F(y)$  for  $r = 0$  and  $d_s^\circ < F(y)$  for  $r = s$ . The case  $r = s$  has no sense because the images of all  $(a_j^\circ, b_j^\circ)$  s are less than  $d_s^\circ$ . When  $r = 0$  since  $F(\beta_0^\circ) = F(x_{i-1}) \leq c_1^\circ < F(y)$ , the intermediate value theorem implies that there exists a point  $t \in [x_{i-1}, y)$  such that  $F(t) = c_1^\circ$ . But according to the definition of  $\alpha_1^\circ$  we must have  $\alpha_1^\circ \leq t$  which is in contradiction with the position of  $y$ .

Finally when  $1 \leq r \leq s-1$ , since  $F(\beta_r^\circ) = d_r^\circ < c_{r+1}^\circ < F(y)$ , the intermediate value theorem implies that there exists a point  $t' \in (\beta_r^\circ, y)$  such that  $F(t') = c_{r+1}^\circ$  but according to the definition of  $\alpha_{r+1}^\circ$  we must have  $\alpha_{r+1}^\circ \leq t'$  which is a contradiction. Thus, our claim is proved, that is, non of the points  $a_j^\circ$  or  $b_j^\circ$  lie inside the intervals  $(\beta_r^\circ, \alpha_{r+1}^\circ)$  or in another words, all points  $a_j^\circ$  and  $b_j^\circ$  lie only inside intervals  $[\alpha_r^\circ, \beta_r^\circ]$ .

The above fact admits the definition of partition  $S_2$  as

$$(4.10) \quad \begin{aligned} S_2 : x_{i-1} &= \beta_0^\circ \leq \alpha_1^\circ \leq a_1^\circ < b_1^\circ < \dots < a_{j_1}^\circ < b_{j_1}^\circ \leq \beta_1^\circ \\ &< \alpha_2^\circ \leq a_{j_1+1}^\circ < b_{j_1+1}^\circ < \dots < a_{j_2}^\circ < b_{j_2}^\circ \leq \beta_2^\circ \\ &< \alpha_3^\circ < \dots < \alpha_s^\circ \leq \dots < a_m^\circ < b_m^\circ \leq \beta_s^\circ \leq \alpha_{s+1}^\circ = x_i, \end{aligned}$$

which is clearly a refinement of partition  $S_1$  defined in (4.9). Thus, according to our hypothesis we see that  $|F(S_2)| - |F(S_1)| < \epsilon$  which by a simple but careful observation results in the following relation

$$\sum_{j=1}^m |F(b_j^\circ) - F(a_j^\circ)| < \epsilon + \sum_{r=1}^s |F(\beta_r^\circ) - F(\alpha_r^\circ)|.$$

Recalling the definitions of  $\alpha_r^\circ$  and  $\beta_r^\circ$  and since  $J_{k_r}^\circ = (c_r^\circ, d_r^\circ)$ , the above relation converts to

$$\sum_{j=1}^m |F(b_j^\circ) - F(a_j^\circ)| < \epsilon + \sum_{r=1}^s \lambda(J_{k_r}^\circ),$$

and due to relation (4.1) one obtains

$$\sum_{j=1}^m |F(b_j^\circ) - F(a_j^\circ)| < 2\epsilon.$$

Since the above relation is true for the end points of any finite number (here  $m$ ) of  $I_{kl}^\circ$  s, it is also valid for all of them, that is

$$(4.11) \quad \sum_{k,l} |F(b_{kl}^\circ) - F(a_{kl}^\circ)| < 2\epsilon.$$

Now by gathering the relations (4.6), (4.7) and (4.11) it is found that

$$(4.12) \quad \sum_{k,l=1}^{\infty} |F(b_{kl}) - F(a_{kl})| < 4\epsilon.$$

Inequalities (4.2), (4.4) and (4.12) yield

$$(4.13) \quad \lambda(p(N_i)) \leq \sum_{k,l=1}^{\infty} \lambda(p(I_{kl})) < 5\epsilon.$$

This establishes the zero measure of  $p(N_i)$  when  $F(x_{i-1}) \neq F(x_i)$ .

It only remains to show for the non-decreasing function  $n = p - F$ , that  $\lambda(n(N_i)) = 0$ . In an exactly similar way of obtaining relation (4.2) one easily finds that

$$\lambda(n(N_i)) \leq \sum_{k,l=1}^{\infty} \lambda(n(I_{kl})),$$

where still  $I_{kl} = (a_{kl}, b_{kl})$  and thus  $n(I_{kl}) \subset [n(a_{kl}), n(b_{kl})]$ . Then we notice that for any two points  $x, y \in [x_{i-1}, x_i]$  since  $n = p - F$  we have

$$|n(y) - n(x)| \leq |p(y) - p(x)| + |F(y) - F(x)|.$$

Substituting  $a_{kl}, b_{kl}$  s to resp.  $x, y$  the latter relation yields

$$\lambda(n(N_i)) \leq \sum_{k,l=1}^{\infty} \lambda(n(I_{kl})) \leq \sum_{k,l=1}^{\infty} \lambda(p(I_{kl})) + \sum_{k,l=1}^{\infty} |F(b_{kl}) - F(a_{kl})|.$$

The upper bounds for the first and second terms on the right hand side of the above relation due to (4.12) and (4.13) proves the zero measure of  $n(N_i)$ .

## 5. Conclusion

As one little step towards understanding the regularity of hydrodynamical quantities, it was attempted to see a more direct and clear dependence of continuity and integrability through the Lebesgue integral while there is a hope to generalize the method to find the situation for other types of integration. Indeed, there probably exists an alternative kind of absolute continuity in connection with other types of integration rather than the Lebesgue one.

Even further, since the used method here essentially employed the general measure-type informations, it looks to have sense to include the issue of measurability of fluid functions under the mechanism of singularity. In other words, the problem of blow up usually deals with singularities and therefore infinite integrals while it is not yet known if this dynamics can change even the measurability of solutions or not.

It was seen here that the absolute continuity can be extracted directly as a consequence of measure-type properties of functions. There was nowhere used the idea of differentiability which is the result of the Vitali covering lemma. Instead, the Radon–Nikodym theorem was the main tool which relies solely on the excellent consistency between the Lebesgue integral and a measure space.

In addition, Lemma 2.2 was proven showing a slightly more general result than needed for the proof of the Banach-Zarecki theorem. Although the classical version of this theorem was proven here but it is not surprising if one can generalize this proof to more general spaces and even higher dimensions.

### Acknowledgments

The authors wish to sincerely appreciate the referee for useful comments and suggestions which definitely improved the paper.

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